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THE METHOD OF PROJECTION AND DECOMPOSITION OF ANALYTICAL FUNCTIONS IN BOUNDARY-VALUE PROBLEMS OF ELASTICITY THEORY*

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A technique is proposed for reducing boundary-value problems of elasticity theory in multiply connected regions to a system of algebraic equations. The technique is based on the projection method for analytical functions of a complex variable combined with decomposition of the original region. The starting equations are provided by the Laurent series expansion of the necessary and sufficient condition of analyticity of functions. The coordinate functions are the terms of the Laurent series for the required potentials of elasticity theory in each of the subregions obtained from the original region by decomposition. The proposed method avoids the construction of integral equations, while preserving the advantages of the boundary-element method.

1. ***Analyticity conditions.*** The necessary and sufficient condition for the function Φ to be analytic in a given region B with the boundary ∂B may be represented in the form /1/

$$\int_{\partial B} \frac{\Phi(t)\,dt}{t-z} = 0, \quad z \notin B \tag{1.1}$$

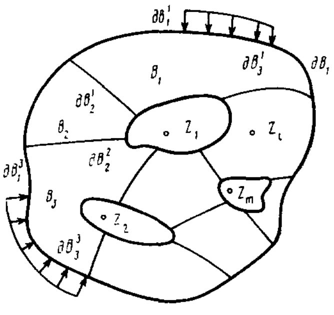

Fig.1

We assume that B is an arbitrary, closed, multiply connected region whose boundary ∂B satisfies the Hölder conditon (Fig.1), and the point at infinity does not belong to B. If $\{z_m\}$ are arbitrary points of the interior subregions that do not belong to B, then condition (1.1), after expansion in a Laurent series, can be replaced by an infinite system of equations for the analytical function Φ,

$$\int_{\partial B} \Phi(t)\,\xi\,dt = 0 \quad (\xi = t^k, (t-z_m)^{-k-1}, k = 0, 1, \ldots) \tag{1.2}$$

Assume that the given region $B = \cup B^i$ is decomposed so that inside each subregion B^i the function $\Phi(z)$ is representable by its Laurent series

$$\Phi^i = \sum_{m=1}^{M} \sum_{s=-1}^{-\infty} \Phi_{ms}^i (z-z_m)^s + \sum_{s=0}^{\infty} \Phi_s^i (z-z_i)^s \tag{1.3}$$

where $M+1$ is the connectivity of the region B, z_i is an arbitrary point of the subregion B^i. Analytical continuity conditions for Φ should be satisfied on the joining curves of the subregions.

The functions in the expansion (1.3) are selected as the coordinate functions of the

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projection method.

2. ***Mixed problem of elasticity theory.*** The boundary conditions of the mixed problem of elasticity theory can be represented in terms of complex potentials in the form /2/

$$\varphi - \beta t\overline{\varphi}' - \beta\overline{\psi} = \alpha u, \quad \alpha = 2G\beta \tag{2.1}$$

for the parts of the boundary on which the displacements are given and

$$\varphi + t\overline{\varphi}' + \overline{\psi} = f, \quad f = i\int_{t_1}^{t} (X_n + iY_n)\,dt + C_1 \tag{2.2}$$

for the parts of the boundary on which the stresses are given. Here φ and ψ are the required analytic functions of a complex variable, t are the boundary coordinates, $\beta = (3 - 4\sigma)^{-1}$ for the plane strain problem, $\beta = (1 + \sigma)(3 - \sigma)^{-1}$ for the generalized plane stress state, G is the shear modulus, u is the complex displacement of the boundary points, and $X_n + iY_n$ is the external load applied to the points of the boundary of B.

The following conditions should be satisfied on the interface curves:

$$u^i = u^j, \quad (X_n + iY_n)^i = (X_n + iY_n)^j \tag{2.3}$$

where the superscripts i and j identify the adjoining subregions.

As we know /2/, $X_n + iY_n = -id(\varphi + t\overline{\varphi}' + \overline{\psi})/dt$, so that the second equation in (2.3) can be rewritten as

$$(\varphi + t\overline{\varphi}' + \overline{\psi})^i = (\varphi + t\overline{\varphi}' + \overline{\psi})^j + C_2^* \tag{2.4}$$

Using (2.1), we can rewrite the first equation in (2.3) in the form

$$(\alpha^{-1}(\varphi - \beta t\overline{\varphi}' - \beta\overline{\psi}))^i = (\alpha^{-1}(\varphi - \beta t\overline{\varphi}' - \beta\overline{\psi}))^j \tag{2.5}$$

From (2.4) and (2.5) we now obtain the formula

$$\varphi^i = a\varphi^j + bt\overline{\varphi}^{j\prime} + b\overline{\psi}^j + C_2, \quad C_2 = C_2^*\beta^i(1 + \beta^i)^{-1} \tag{2.6}$$

$$a = (\beta^i/\alpha^i + 1/\alpha^j)/(\beta^i/\alpha^i + 1/\alpha^i), \quad b = (\beta^i/\alpha^i - \beta^j/\alpha^j)/(\beta^i/\alpha^i + 1/\alpha^i)$$

Projecting Eqs.(2.1), (2.2), and (2.6) on to Eqs.(1.2), we obtain

$$\int\limits_{\partial B_1^i} (-\beta^i t\overline{\varphi}^i - \beta^i\overline{\psi}^i)\,\xi\,dt + \int\limits_{\partial B_2^i} (t + \overline{\varphi}^{i\prime} + \overline{\psi}^i + C_1)\,\xi\,dt + \tag{2.7}$$

$$\int\limits_{\partial B_3^i} (b^j t\overline{\varphi}^{j\prime} + b^j\overline{\psi}^j + C_2)\,\xi\,dt = \int\limits_{\partial B_1^i} \alpha^i u^i\xi\,dt + \int\limits_{\partial B_2^i} f\xi\,dt$$

$$(\xi = t^k, (t - z_m)^{-k-1};\ i = 1, 2, \ldots, I;\ k = 0, 1, \ldots;\ m = 1, 2, \ \ldots, k, M)$$

Changing to conjugate variables in Eqs.(2.1), (2.2), and (2.6) and again projecting the conjugate equations on to Eqs.(1.2), we obtain a second group of equations

$$\int\limits_{\partial B_1^i} (\alpha^i\overline{\varphi}^i - \beta^i\bar{t}\varphi^{i\prime})\,\xi\,dt + \int\limits_{\partial B_2^i} (\overline{\varphi}^i + \bar{t}\varphi^{i\prime} + C_1)\,\xi\,dt + \tag{2.8}$$

$$\int\limits_{\partial B_3^i} (a\overline{\varphi}^j + b\bar{t}\varphi^{j\prime} + \overline{C}_2)\,\xi\,dt = \int\limits_{\partial B_1^i} \alpha^i\bar{u}^i\xi\,dt + \int\limits_{\partial B_2^i} \bar{f}\xi\,dt$$

$$(\xi = t^k, (t - z_m)^{-k-1};\ i = 1, 2, \ldots, I;\ k = 0,\ 1, \ldots,\ k;\ m = 1, 2, \ldots M)$$

Here ∂B_1^i is the part of the boundary with given stresses, ∂B_2^i is the part of the boundary with given displacements, ∂B_3^i is the interface curve between subregions B^i and B^j, and I is the number of subregions in region B.

Replacing the boundary values of the analytical functions φ and ψ with their corresponding reduced Laurent series expansions, we transform Eqs.(2.7) into a linear system of algebraic equations. Let

$$\varphi^i = \sum_{m=1}^{M}\sum_{s=-1}^{N-1} \varphi_{sm}^i (z - z_m)^s + \sum_{s=0}^{N_2} \varphi_s (z - z_i)^s \tag{2.9}$$

$$\psi^i = \sum_{m=1}^{M}\sum_{s=-1}^{-N_1} \psi_{sm}^i (z - z_m)^s + \sum_{s=0}^{N_2} \psi_s (z - z_i)^s$$

$$\varphi^{i\prime} = \sum_{m=1}^{M}\sum_{s=-1}^{-N_1} s\varphi_{sm}^i (z - z_m)^{s-1} + \sum_{s=0}^{N_2} s\varphi_s (z - z_i)^{s-1}$$

Substituting the representations (2.9) into Eqs.(2.7) and (2.8), we obtain the linear systems of algebraic equations

$$TA = F \tag{2.10}$$

where A is the vector of the required expansion coefficients in (2.9), and T and F, respectively, are the matrix and the vector whose components, apart from a constant, are a composition of integrals of the form

$$\int_{\partial B^i} s(t - z_m)^{-s+1} t^{k+1}\,dt, \quad \int_{\partial B^i} st\,(\overline{t - z_m})^{-s+1}(t - z_m)^{-k}\,dt, \tag{2.11}$$
$$\int_{\partial B^i} (\overline{t - z_n})^{-s}(t - z_m)^{-k}\,dt, \quad \int_{\partial B^i} t^{k+1}(\overline{t - z_i})^{s}\,dt$$

etc. Here s is an integral exponent with values between the bounds of the reduced Laurent series, $\{s: -N_1, N_2\}$, and k is an integral exponent bounded by the number of equations in (2.7) and (2.8). In order to evaluate integrals of the form (2.11), it is better to use the equality $dt = -indl$, where n is the outer unit normal to the boundary, and dl is an element of length of the boundary. In order to obtain a single-valued solution of Eqs.(2.7) and (2.8), it is necessary that $C_1 = 0$ or $C_2 = 0$ and also

$$(MN_1 + N_2 + 1) = K$$

The components of the displacement vector in our problem take the form

$$u = \sum_{m=1}^{M}\sum_{s=-1}^{-N_1} (\alpha^i\varphi_{sm}^i (z - z_m)^s - (\beta^i z\overline{\varphi}_{sm}^i (s + 1) - \beta\overline{\psi}_{sm})(\overline{z - z_m})^s) + \tag{2.12}$$
$$\sum_{s=0}^{N_2} (\alpha^i\varphi_s (z - z_i)^s - (\beta^i z\overline{\varphi}_s{}^i (s + 1) - \beta\overline{\psi}_s)(z - z_i)^s)$$

and the components of the stress field take the form

$$\sigma_X + \sigma_Y = 4\,\mathrm{Re}\,\Big(\sum_{m=1}^{M}\sum_{s=-1}^{-N_1} s\varphi_{sm}^i (z - z_m)^{s-1} + \sum_{s=0}^{N_2} s\varphi_s (z - z_i)^{s-1}\Big) \tag{2.13}$$
$$\sigma_Y - \sigma_X + 2i\tau_{XY} = 2\,\Big(z\sum_{m=1}^{M}\sum_{s=-1}^{-N_1} (s(s - 1)\varphi_{sm}(z - z_m)^{s-2} +$$
$$s\psi_{sm}(z - z_m)^{s-1}) + \sum_{s=0}^{N_2} (s(s - 1)\varphi_s(z - z_i)^{s-2} + s\psi_s(z - z_i)^{s-1})\Big)$$

3. ***Torsion of prismatic rods.*** Introducing the torsion function φ, we reduce our problem to a Laplace equation. The shear stresses are related to the torsion function by the formulas

$$\tau_{XZ} - i\tau_{YZ} = \theta G\,((\partial\varphi/\partial X - i\partial\varphi/\partial Y) - i\bar{z}) \tag{3.1}$$

where θ is the torsion angle, and G is the shear modulus.

We introduce the function

$$\Phi = \partial\varphi/\partial X - i\partial\varphi/\partial Y = 2\partial\varphi/\partial z$$

which is analytical, because

$$\partial\Phi/\partial\bar{z} = 2\partial^2\varphi/\partial z\partial\bar{z} = 4\nabla^2\varphi = 0$$

Formula (3.1) now takes the form

$$\tau_{XZ} - i\tau_{YZ} = \theta G\,(\Phi - i\bar{z}) \tag{3.2}$$

On the unstressed boundary ∂B of the region B, the boundary condition of the torsion problem is expressed in our notation by

$$\mathrm{Re}\,((\tau_{XZ} - i\tau_{YZ})\,n) = \theta G\,\mathrm{Re}\,((\Phi - i\bar{z})\,n) = 0$$

or

$$\Phi n + \overline{\Phi n} = -i\,(z\bar{n} - \bar{z}n) \tag{3.3}$$

If the region B is decomposed so that $B = \cup B^i$, then on the interface curves of the subregions B^i and B^j we should have the conditions of continuity of the torsion function

$$\varphi^i = \varphi^j \tag{3.4}$$

and equality of the tangential pressures

$$\mathrm{Re}\,((\tau_{XZ} - i\tau_{YZ})\,n)^i = \mathrm{Re}\,((\tau_{XZ} - i\tau_{YZ})\,n)^j \tag{3.5}$$

Differentiating Eq.(3.4) along the interface curve with respect to the tangent to the curve, we obtain

$$d\varphi^i\,/\,ds = d\varphi^j\,/\,ds$$

or

$$(\mathrm{Re}\,((\partial\varphi/\partial X - i\partial\varphi/\partial Y)\,\tau))^i = (\mathrm{Re}\,((\partial\varphi/\partial X - i\partial\varphi/\partial Y)\,\tau))^j$$

where τ is the unit tangent to the boundary. Hence, noting that $\tau = -in$, we obtain

$$\mathrm{Im}\,(\Phi n)^i = \mathrm{Im}\,(\Phi n)^j \tag{3.6}$$

Using (3.2), we rewrite Eq.(3.5) in the form

$$G^i\,\mathrm{Re}\,(\Phi n)^i - G^j\,\mathrm{Re}\,(\Phi n)^j = -\,(G^i - G^j)\,\mathrm{Im}\,(\bar{z}n) \tag{3.7}$$

Thus, Eqs.(3.6) and (3.7) imply that the following condition should be satisfied on the interface curve:

$$\Phi^i n - \frac{G^i+G^j}{G^i}\,\Phi^j n + \frac{G^i-G^j}{2G^i}\,\Phi^j n = -\,\frac{G^i-G^j}{G^i}\,\mathrm{Im}\,(\bar{t}n) \tag{3.8}$$

Here and above, n is the outer normal to the joining curve relative to the subregion B^i.

Substituting Eqs.(3.3) and (3.8) into conditions (1.2) for each subregion B^i and noting that $dt = -ndl$, we obtain the system of equations

$$\int\limits_{\partial B_1{}^i} \overline{\Phi}{}^i n\xi\,dl + \int\limits_{\partial B_2{}^i}\Big(-\frac{G^i+G^j}{2G^i}\,\Phi^j n + \frac{G^i-G^j}{2G^i}\,\overline{\Phi}{}^j n\Big)\xi\,dl = \tag{3.9}$$

$$-\,i\int\limits_{\partial B_1{}^i}(t\bar{n} - \bar{t}n)\,\xi\,dl - \frac{G^i-G^j}{2G^i}\int\limits_{\partial B_2{}^i}(t\bar{n} - \bar{t}n)\,\xi\,dl$$

$$(\xi = t^k,\,(t - z_m)^{-k-1};\ k = 0, 1, \ldots)$$

Here $\partial B_1{}^i$ is the free boundary of the subregion B^i, and $\partial B_2{}^i$ is the interface boundary between subregions B^i and B^j.

If the function Φ^i is approximated by the coordinate functions

$$\Phi^i = \sum_{m=1}^{M}\sum_{s=-1}^{-N_1}\Phi_{sm}\,(z - z_m)^s + \sum_{s=0}^{N_2}\Phi_s\,(z - z_i)^s \tag{3.10}$$

then substituting the representation (3.10) into Eq.(3.9) we obtain a linear system of equations $A\Phi = f$, where the elements of the matrix A and the vector f are given by (2.11).

The shear stresses can be computed from the formula

$$(\tau_{XZ} - i\tau_{YZ})^i = \theta G^i\,\Big(\sum_{m=1}^{M}\sum_{s=-1}^{-N_1}\Phi_{sm}\,(z - z_m)^s + \sum_{s=0}^{N_2}\Phi_s\,(z - z_i)^s - i\bar{z}\Big)$$

The torsional rigidity of the bar is determined from the formula

$$D = -\int\limits_B \mathrm{Im}\,((\tau_{XZ} - i\tau_{YZ})\,\bar{z}\,dX\,dY = \tag{3.11}$$

$$\frac{1}{2}\,\mathrm{Im}\sum_i G^i\int\limits_{\partial B^i}\Big(-\,\Phi^i\,\frac{\bar{z}^2}{3} + i\,\frac{\bar{z}^3}{3}\Big)\,n\,dl$$

where, by the analyticity of the function Φ, the integral over the region has been replaced with a line integral over the boundary.

4. *Bending of composite rods by a transversal force.* Following Muskhelishvili /2/, the problem of the bending of beams by a transversal force with normal stresses

$$\sigma_z = -W_X\,(l - s)\,X/I_Y,\quad \sigma_X = \sigma_Y = 0 \tag{4.1}$$

and zero shear stresses τ_{XY} can be reduced to the plane problem of elasticity theory. Here σ_t are the corresponding normal stresses, W_X is the transverse force parallel to the X-axis,

l is the bar length, s is the coordinate along the rod axis, and I_Y is the moment of inertia of the bar cross-section about the Y-axis.

For cross-sections made of materials with different Poisson's ratios, the shear stresses can be represented in complex form as

$$(\tau_{XZ} + i\tau_{YZ})_k = -W_l G^i I_l^{-1}(\overline{X}_k + \overline{X}_{0k} + g_k(z) - U_k) \tag{4.2}$$
$$g_k(z) = i^{k-1}\left(\left(\frac{\sigma}{2} + \frac{1}{4}\right)z^2 - \frac{3}{4}\bar{z}^2 + \frac{z\bar{z}}{(-1)^k 2}\right)$$
$$k,\ l = 1, 2,\ k \neq l$$

The first formula ($k = 1, l = 2$) is written for the case of a transverse force parallel to the X axis, and the second ($k = 2, l = 1$) for the case of a transverse force parallel to the Y axis. Here X_k is the required analytical function, U_k are the displacements obtained by solving the so-called supplementary problems of elasticity theory /2/, X_0 is a particular solution of the Poisson equation, σ is Poisson's ratio, and

$$4\partial^2\chi_{0k}/\partial z\partial\bar{z} = (\partial U_k/\partial z + \partial U_k/\partial\bar{z})(1 - 2\sigma)^{-1} \tag{4.3}$$

Since

$$\partial U_k/\partial z + \partial U_k/\partial\bar{z} = (1 - \beta)\,\alpha^{-1}(\varphi_k' + \bar{\varphi}_k') \tag{4.4}$$

where φ is the complex potential of the supplementary problems, we have

$$X_{0k} = (1 - \beta)(1 - 2\sigma)^{-1}(\varphi_k' z + \varphi_k)/4 \tag{4.5}$$

These representations enable us to reduce our problem to a boundary-value problem for an analytic function with the following boundary conditions:

on the free boundaries

$$X_k n + \overline{X_k n} = f_k \tag{4.6}$$

on the interface curves

$$G^i(X_k n + \overline{X_k n})^i - G^j(X_k n + \overline{X_k n})^j = G^i f^i - G^j f^j \tag{4.7}$$
$$(X_k n - \overline{X_k n})^i - (X_k n - \overline{X_k n})^j = 0$$

Here

$$-f_k = \overline{X}_{0k} + g_k(z) - U_k$$

Thus, the problem with boundary conditions (4.6) and (4.7) for the analytical function X has the same form as the torsion problem (3.3) and (3.7) for the function Φ, differing only by the right-hand side of Eqs.(3.8), which in the present case equals $-\operatorname{Im}((G^i f_k^i - G^j f_k^j)\,n)$ The values of this expression can be found by solving the supplementary problems of elasticity theory using the equations of Sect.1.

The solution of problem (4.6), (4.7), obtained in the same way as the solution of the torsion problem, gives the coordinates of the centre of bending of the cross-section (x_k) from the formula

$$x_k = \int_B \operatorname{Im}((\tau_{XZ} + i\tau_{YZ})_l\,\bar{z}\,dX\,dY \tag{4.8}$$

where $(\tau_{XZ} + i\tau_{YZ})_l$ are the shear stresses produced by the transverse force along the X_l axis. Substituting (4.2) into (4.8) and applying the formula for passing from an area integral to a line integral

$$\int_B f(z,\bar{z})\,dX\,dY = \frac{1}{2}\int_{\partial B}\left(\int f(z,\bar{z})\,d\bar{z}\right) n\,dl$$

we obtain

$$x_k = (2I_k)^{-1}\sum_i G^i\Big(\int_{\partial B^i} X_k z\bar{z} + ((1-\beta)(1-2\sigma)^{-1}\alpha^{-1}/4 + \tag{4.9}$$
$$(1-\beta)\alpha^{-1})\,\varphi_k' z\bar{z}^2/2 + ((1-\beta)(1-2\sigma)^{-1}\alpha^{-1} - \alpha^{-1})\,\varphi_k z\bar{z} -$$
$$(1-\beta)\,\alpha^{-1}\psi_k\bar{z}^2/2 + i^{k-1}\left(\left(\frac{\sigma}{2} + {}^1/_4\right)z^2\bar{z}^2/2 - 3\bar{z}^4/16 + (-1)^k z\bar{z}^3/6\right) n\,dl\Big)$$

The solutions of problems (4.6), (4.7) enable us to compute the directional shear rigidities of the region from the formulas /3/

$$(GF)_k = k_k GF = W_k^2 / J_{k2} \quad J_k = \int_B |\tau_{XZ} + i\tau_{YZ}|_k^2 \, dX\, dY$$

whence, substituting (4.2), we obtain

$$(GF)_k = I_l^2 \Big(\sum_i G^i \int_B |\overline{X}_k + \overline{X}_{0k} + g_k(z) - U_k|^2 \, dX\, dY\Big) \tag{4.10}$$

5. ***A bound of the method.*** *Theorem.* Let the boundary condition

$$\Phi = F(\overline{\Phi}, \Phi', \overline{\Phi}', t) \tag{5.1}$$

be given for the analytic function Φ defined in the multiply connected region B with the boundary ∂B that satisfies the Hölder condition. The function F satisfies the Hölder condition

$$|F(t) - F(t_0)| \leqslant A\, |t - t_0|^{\mu}, \quad \mu < 1$$

uniformly in Φ. Let

$$\Delta = \Phi^* - F(\overline{\Phi}^*, \Phi^{*\prime}, \overline{\Phi}^{*\prime}, t) \tag{5.2}$$

be the boundary value of a function for which Φ^* is an arbitrary analytical function in the region B.

If the following systems of equalities holds:

$$\int_{\partial B} F\xi\, dt = 0 \quad (\xi = (t^k, k = 0, 1, \ldots, K), \xi = (t - z_m)^k, \\ k = -1, \ldots, -N;\ m = 1, \ldots, M) \tag{5.3}$$

where $\{z_m\}$ are arbitrary points of the interior simply connected regions that do not belong to the region B, then there exists a decomposition of B such that

$$\Delta = o(\min_m (|t|^{-K-2}, |t - z_m|^N)) \tag{5.4}$$

Proof. Consider the function

$$\varphi = \int_{\partial B} \Delta\, (t - z)^{-1}\, dt \tag{5.5}$$

which is analytic at all points $z \notin B$. Since the function Φ^* is analytic in the region B, we have

$$\varphi = -\int_{\partial B} F(\overline{\Phi}^*, \Phi^{*\prime}, \overline{\Phi}^{*\prime}, t)\,(t - z)^{-1}\, dt \tag{5.6}$$

By Laurent's theorem on the expansion of an analytical function outside the region B, we find

$$\varphi = \sum_{m=1}^{M} \sum_{s=1}^{\infty} \varphi_{ms} (z - z_m)^s - \sum_{s=0}^{\infty} \varphi_s z^{-s} \tag{5.7}$$

where φ_{ms} and φ_s are the expansion coefficients of the function φ.

By the Sokhotskii-Plemel' formula /1/, using (5.3), we obtain for the points $t \in dB$

$$\frac{1}{2}\Delta + \frac{1}{2\pi i}\int_{\partial B} \Delta\,(t - t_0)^{-1}\, dt = \sum_{m=1}^{M} \sum_{s=N+1}^{\infty} \varphi_{ms} (t - z_m)^s + \sum_{s=K+2}^{\infty} \varphi_s t^{-s}$$

whence

$$\Delta \leqslant |(2\pi\iota)^{-1} \int_{\partial B} |(F(t) - F(t_0))(t - t_0)^{-\mu}| \, |t - t_0|^{\mu - 1} d|t - t_0| + \Sigma_1 + \Sigma_2 \leqslant$$
$$(2\pi\mu)^{-1} AR^{\mu} + |\Sigma_1| + |\Sigma_2| \tag{5.8}$$

Here $R = \max(t - t_0)$ is the maximum distance between the points in the decomposed region. If

$$R \leqslant (2\pi\mu)^{\mu} A^{-\mu} o \,(\min_m (|t - z_m|^N, |t|^{-K-2})) \tag{5.9}$$

then inequality (5.8) gives the bound (5.4). The theorem is proved.

6. ***Numerical implementation.*** This study has provided the basis for a package of programs, whose capabilities are demonstrated here by comparing the calculated and published values of the geometrical and rigidity characteristics of a number of test sections (Figs.2 and 3) and by reporting some new results on the characteristics of the blade profiles of wind-power installations (Figs.4 and 5).

Fig.2 is a cross-section in the shape of a square ring 0.01 m thick and 0.10 m on the external side; the ring has a cut in one of the sides. The coordinates of the centre of bending calculated from standard formulas /3/ for thin-walled elements and from formula (4.9) were found to be -0.112 m and -0.106 m, respectively; the torsional rigidity is 4.73×10^5 Nm^2 according to the data of /4/ and 4.71×10^5 Nm^2 by formula (3.11). We also used formula (4.10) to find the directional shear rigidities $(GF)_X = 228.5$ N, $(GF)_Y = 175.6$ N. The shear modulus was $G = 3.98 \times 10^4$ Nm^2. For the cross-section shown in Fig.3 /4/, with the same shear modulus, the torsional rigidity was 1.17×10^6 Nm^2 by both formulas, and the directional shear rigidities were $(GF)_X = 527.8$ N and $(GF)_Y = 288.4$ N. For this cross-section we also calculated the shear stresses at the interior points of the region. The graphical comparison of the numerical results shown in Fig.3 indicates that the data in /4/ are close to our data.

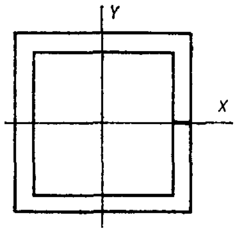

Fig.2

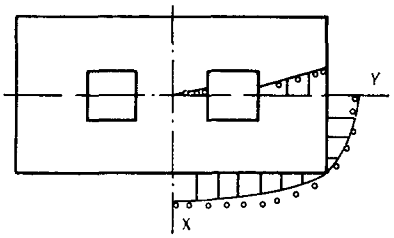

Fig.3

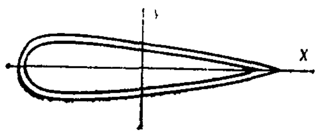

Fig.4

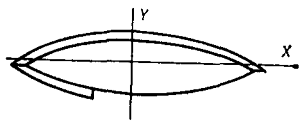

Fig.5

Figs.4 and 5 show the blade cross-sections of wind-power installations. The chord is B=0.30 m, the height is 0.072 m and 0.064 m, the wall thickness is 0.004 m and 0.006 m, respectively. For a shear modulus $G = 2.65 \times 10^{11}$ Nm^2, the coordinates of the centre of bending for the cross-section shown in Fig.4 are $x = -0.0399$ m, $y = 0$, the directional shear rigidities are $(GF)_X = 1.28 \times 10^8$ N, $(GF)_Y = 9.39 \times 10^7$ N, the torsional rigidity is D=1.63×10^5 Nm^2. For the cross-section shown in Fig.5, the corresponding figures are $x = -0.194$ m, y=0.0293 m, $(GF)_X = 1.21 \times 10^8$ N, $(GF)_Y = 7.03 \times 10^7$ m, D=1.83×10^4 Nm^2. In all the cross-sections shown, the origin of the coordinates is at the centre of mass.

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